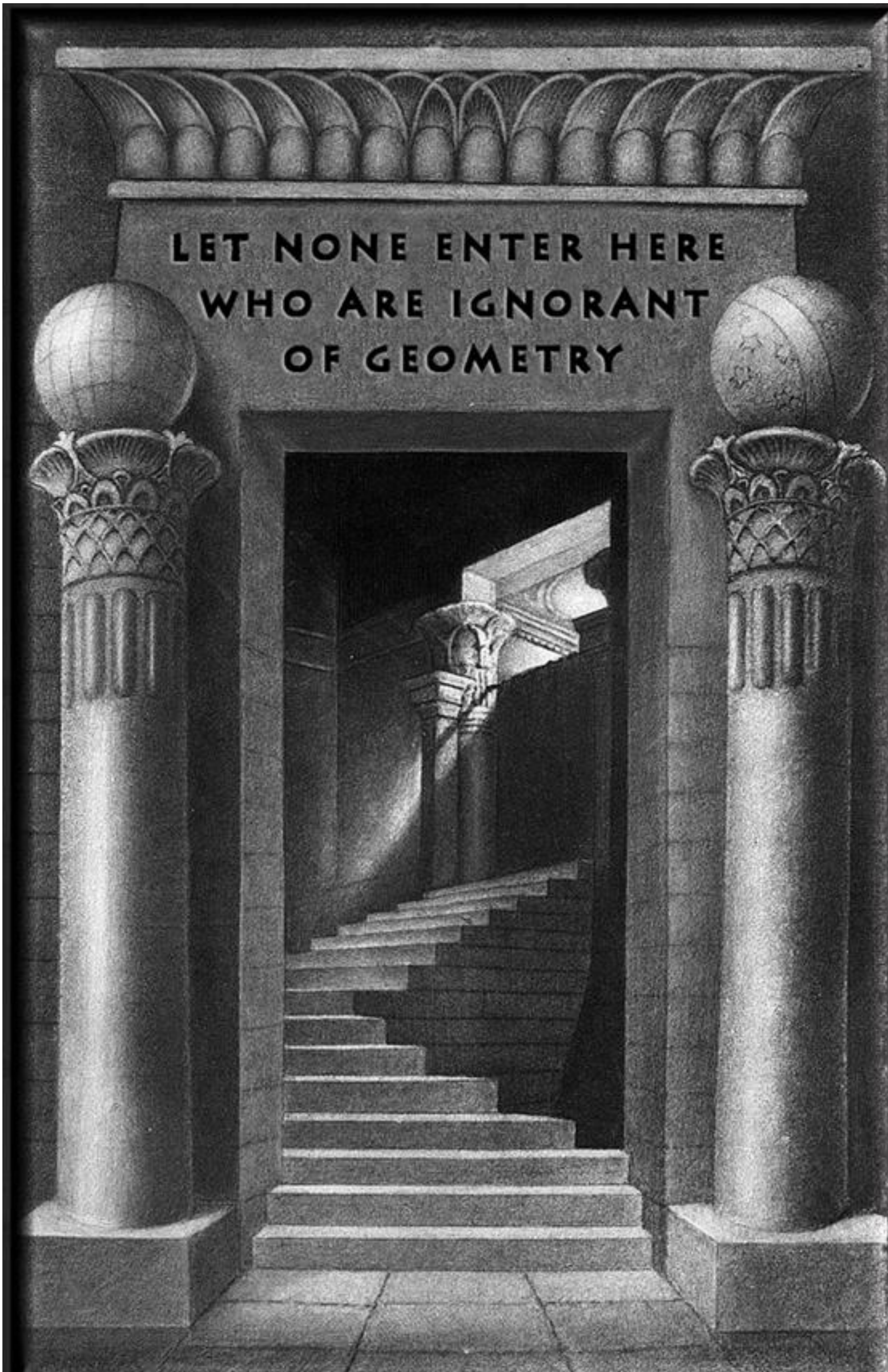

LET NONE ENTER HERE
WHO ARE IGNORANT
OF GEOMETRY

Geometry: Some Definitions and Basic Information

Angle: Two straight lines drawn from the same point, and diverging from each other, form an opening called an angle. An angle can be expressed by three letters with the middle letter representing the vertex, or point of divergence. A single letter corresponding to the vertex can also express an angle, or a letter placed within the angle such as "x" or, frequently a Greek letter can serve the same purpose. The **magnitude** of an angle does not depend upon the length of the lines forming the angle, rather it depends upon the distance between them, that is, the amount of divergence separating one line from the other. The number of degrees of arc that the sides of the angle intersect, assuming the center of the arc is placed at the vertex of the angle, measures the magnitude of an angle. Angles are of three varieties: A right angle is an angle where the two lines are perpendicular, or 90º, an obtuse angle contains more than 90º and an acute angle contains less than 90º. An angle whose vertex is at the center of a circle is called a **central angle**. An angle drawn within a circle whose vertex fall on the circles circumference is called an **inscribed angle**.

Perpendicular: One line is perpendicular to another when the two angles formed by the two lines are equal in magnitude. When using the sexigesimal system of circle measurement if follows that the measure of the angles so formed would be 90º.

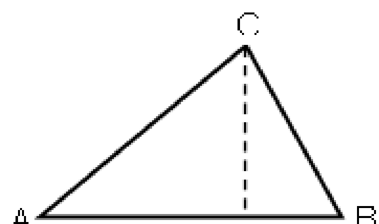
Parallel: Two lines are parallel if all perpendiculars drawn from one line to the other are equal in length. It follows that if all perpendiculars drawn from one line to another line are equal in length the two lines must be perpendicular.

Two geometric figures are said to be equal if they enclose an equal amount of space, or area. A circle and triangle, for example will be equal if the area enclosed by each is the same.

Two geometric figures are said to be **identical**, or **congruent**, if they are equal in all their respective parts, that is all angles and sides are equal in measure and they enclose exactly the same area. Two identical figures will exactly superimpose with perfect coverage of one over the other. Identical geometric figures are both the same size and the same shape.

Two geometric figures are said to be **similar** if they are of the same shape but different size, that is, all the angles of one are equal to all the corresponding angles of the other.

A **triangle** is a figure composed of three straight lines. If all three lines are the same length the triangle is said to be **equilateral.** If two sides are the same length and the third side different the triangle is called **isosceles**. If all three sides are of different length the triangle is said to be **scalene**. A triangle with a right angle is called a **right triangle**. A right triangle can be either isosceles or scalene.



It is common practice to draw a triangle with one side horizontal, as side AB in the triangle above. This side, which is below the rest of the triangle, is called its **base**. The vertex opposite the base is called the **apex** of the triangle. Also, in the above triangle the side CD, drawn perpendicular to the base up to the apex, is called the **altitude**. The length of this line is frequently thought of as the height of the triangle. Any side of the triangle can serve as the base. That being the case a triangle can have three possible altitudes. A general definition for altitude would then be: An altitude of a triangle is a line drawn from any vertex perpendicular to the opposite side.

A circle can be thought of as the set (or locus) of points equidistant from a fixed point called the **center**. The set of points so designated is called the **circumference** of the circle. Any line connecting the center of the circle to its circumference is called a radius of the circle. By definition, all radii of a given circle must be equal in length. A circle can have an infinite number of

The Propositions of Euclidian Geometry

1. All radii of the same circle are equal.
2. The construction of the equilateral triangle by means of the vesica.
3. Triangles which have two sides and the angle contained by them equal are identical. (SAS)
4. In an isosceles triangle the angles at the base are equal.
5. Triangles which have all three sides equal are identical. (SSS)
6. To divide a straight line into two equal parts.
7. From a given point <u>not</u> on a straight line to draw a perpendicular to that line.
8. From a given point <u>on</u> a straight line to draw a perpendicular to that line.
9. The diameter of a circle divides the circumference into two equal parts.
10. A straight line which meets another straight-line forms with it two angles which together are equal to two right angles.
11. A straight line drawn perpendicular to another straight line makes right angles with it.
12. If two straight lines intersect the vertical, or opposite, angles are equal.
13. If a line is perpendicular to one of two parallel lines, it is also perpendicular to the other.
14. If one line is perpendicular to two other lines, these two lines are parallel.
15. The opposite sides of a rectangle are parallel.
16. The opposite sides of a rectangle are equal.
17. A straight line falling upon parallel lines makes the alternate angles equal.
    Corollary: In identical triangles the equal angles are always opposite equal sides.
18. If one straight line falling upon two other lines makes the alternate angles equal, these two lines are parallel.
19. If one line falls upon two parallel lines, it makes the interior angle equal to the exterior.
20. If one line falling upon two other lines makes the internal angle equal to the external, those two lines are parallel.

21. Through a given point to draw a parallel to a given line.

22. The three angles of a triangle taken together are equal to two right angles.
    Corollary: If two angles of any triangle are known, the third is also known, for it is that which the other two together require to be equal to two right angles.

23. If two triangles have two angles equal, they have also the third angle equal.

24. The exterior angle of any triangle is equal to the two interior and opposite angles taken together.

25. Triangles which have two angles and the side which lies between them equal are identical. (ASA)

26. If two angles of a triangle are equal, the sides opposite to those angles are equal.

27. The opposite sides of a parallelogram are equal.

28. Parallelograms which are between the same parallels, and have the same base, are equal.

29. If a triangle and a parallelogram are upon the same base, and between the same parallels, the triangle is equal to half the parallelogram

30. Parallelograms which are between the same parallels, and have equal bases are equal.

31. Triangles which are between the same parallels, and have equal bases, are equal.

32. In any right-angled triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides. (The Pythagorean Theorem)

33. To draw the circumference of a circle through three given points.

34. If the radius of a circle bisects a chord, it is perpendicular to that chord.
    Corollary: The two radii intersecting the endpoints of the chord, form with the given radius two central angles which are also equal.

35. To find the center of a circle

36. To find the center of an arc of a circle.

37. If three equal lines meet in the same point within a circle, and are terminated at that point, they are radii of that circle.

38. If the radius of a circle is perpendicular to a chord, the radius bisects both the chord and the arc of the chord.

39. A straight line perpendicular to the extremity of a radius is a tangent to the circle.
Corollary: A perpendicular is the shortest line that can be drawn from any point to a given line.

40. If a straight line is drawn touching a circumference, a radius drawn to the point of contact will be perpendicular to the tangent.

41. The angle formed by a tangent and a chord is measured by half the arc of that chord.

42. An angle at the circumference of a circle is measured by half the arc which it contains (or, by which it is subtended.)

43. The angle at the center of a circle is double the angle at the circumference.

44. Upon a given line, to describe a segment of a circle containing a given angle.

45. In every triangle the greater side is opposite to the greater angle, and the greater angle to the greater side.

46. Two parallel chords intercept equal arcs.

47. If a tangent and a chord are parallel to each other, they intercept equal arcs.

48. The angle formed by the intersection of two chords is measured by half the arcs intercepted by the two chords.

49. The angle formed by two secants is measured by half the difference of the two intercepted arcs.

50. The angle formed by two tangents is measured by half the difference of the two intercepted arcs.
Corollary: By the same logic it can be demonstrated that the angle formed by a tangent and a secant meeting in the same point outside the circle, is measured by half the difference of the two intercepted arcs.

51. To raise a perpendicular at the end of a given line segment.

52. From any point outside of a circle to draw a tangent to that circle.

53. The surface area of a rectangle is equal to the product of its two sides.

54. The surface area of a triangle is equal to half the product of its altitude and its base.

55. To measure the surface of any rectilinear figure.

56. The area of a circle is equal to half the product of its radius and circumference.

57. To draw a triangle equal to a given circle.

58. The area of parallelograms which are between the same parallel lines are to one another as their bases.

59. Triangles which are between the same parallels are to one another as their bases.

60. If a line is drawn in a triangle parallel to one of its sides, it will cut the other two sides proportionately.

61. Equiangular triangles have their homologous (or corresponding) sides proportional.

62. Triangles which have their sides proportional are equiangular. (or similar)

63. Triangles which have an angle in one equal to an angle in the other, and the sides adjacent to these angles proportional, are equiangular. (similar)

64. A straight line which bisects any angle of a triangle divides the side opposite to the bisected angle into two segments, which are proportional to the other two sides.

65. To find a fourth proportional to three given lines.

66. To find a third proportional to two given lines.

67. If four lines be proportional, the rectangle, or product of the extremes is equal to the rectangle or product of the means.

68. Four lines which have the rectangle or product of the extremes equal to the rectangle or product of the means are proportional.

69. If four lines are proportional, they are also proportional alternately.

70. If four lines are proportional, they will be proportional by composition. (Or addition)

71. If four lines be proportional, they will also be proportional by division.

72. If three lines are proportional, the first is to the third as the square of the first is to the square of the second.

73. If two chords in a circle cut each other, the rectangle of the segments of one is equal to the rectangle of the segments of the other.

74. To find a mean proportional between two given lines.

75. The bases and altitudes of equal triangles are in reciprocal or inverse ratio.

76. Triangles which have the bases and altitudes in reciprocal or inverse ratio are equal.

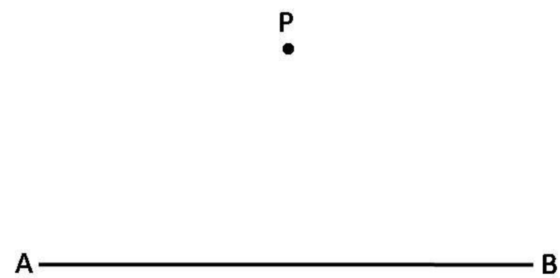
77. Two secants drawn from the same point to a circle are in the inverse ratio of the parts which lie outside the circle.

78. The tangent to a circle is a mean proportional between the secant and the part of the secant which lies outside of the circle.

79. To divide a line in extreme and mean ratio: The Golden Mean.

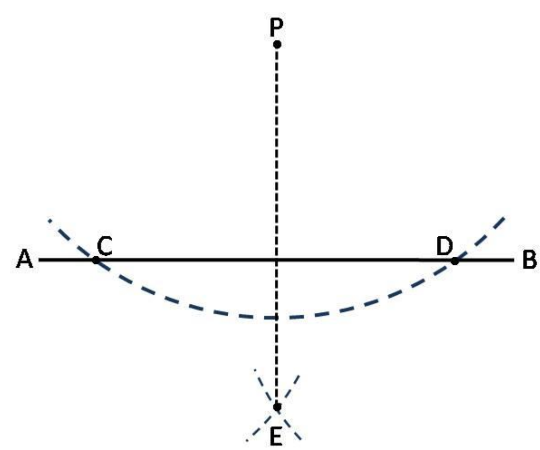
Proposition 7

From a given point not on a straight line to draw a perpendicular to that line.



Given line AB and point P not on the line

Using P as a center draw a circular arc cutting line AB in two points C and D as shown. Using C and D as centers open compass to a radius somewhat greater than half the distance from C to D and draw portions of two arcs of equal radii whose intersection gives point E.



Laying a straight edge from points E to P draw a line of whatever length is appropriate for the problem. That line, EP, will be perpendicular to line AB. The two arcs giving point E can be drawn to intersect either above or below line AB.

radii. Any line passing through the center of a circle that is terminated by the circumference is termed a diameter of the circle. A diameter of a circle must necessarily be twice the length of its radius. A circle can have an infinite number of diameters.

A **quadrilateral** is any figure composed of four straight lines called its sides.

A **parallelogram** is a quadrilateral that has both pairs of opposite sides parallel.

A **rectangle** is a quadrilateral where all four angles are right angles.

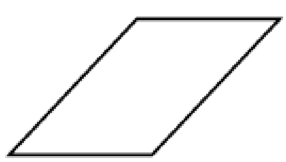
A **square** is a quadrilateral with four right angles and all four sides equal in length.

A **trapezoid** is a quadrilateral figure with one pair of opposite sides parallel. The two parallel sides are referred to as the bases of the trapezoid.

A **trapezium** is a term for any quadrilateral that has no sides parallel.

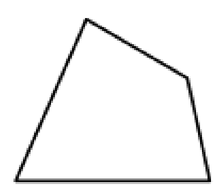
A **rhombus** is a parallelogram with two adjacent sides equal, which, by implication, means that all sides must be equal, making the figure equilateral.

All of the figures below are quadrilaterals

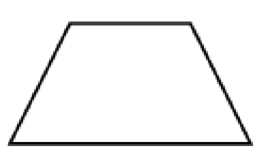


Parallelogram

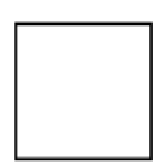
Rectangle



Trapezium

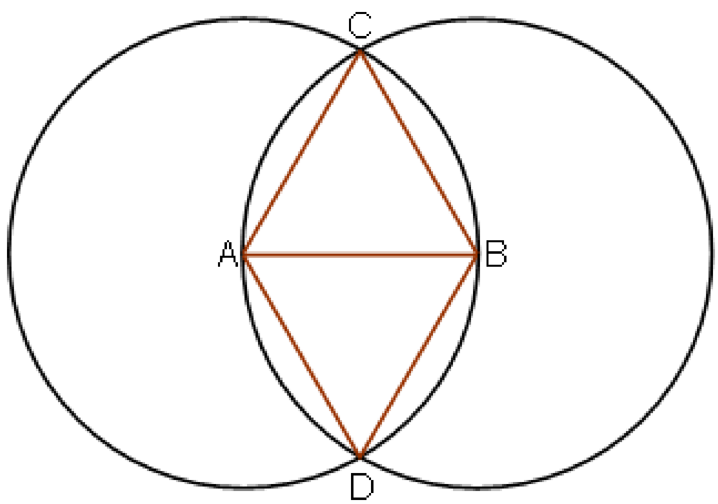


Trapezoid

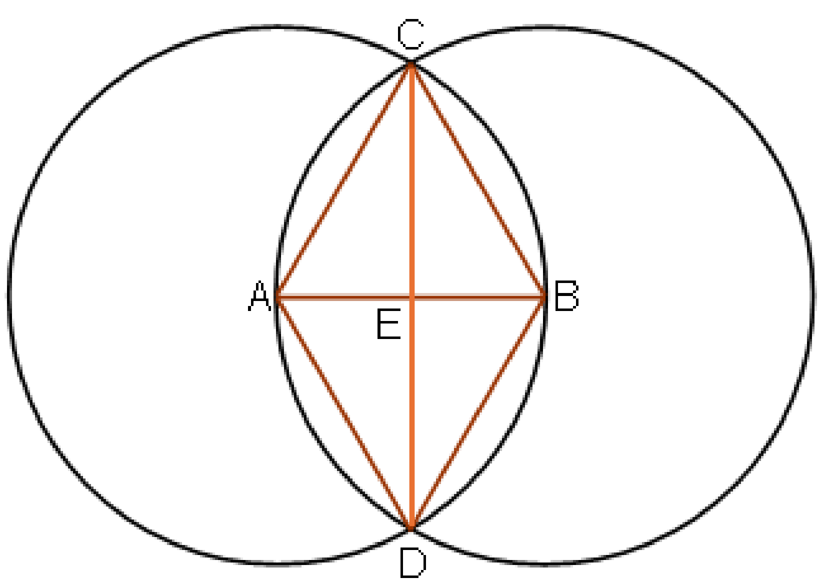


Square

A further development of the Vesica occurs when we insert lines connecting points A to D and B to D, as shown below. This creates the rhombus ACBD, which has the same length to width ratio as the Vesica. We know that lines AD and BD are going to be equal in length to AC and BC.

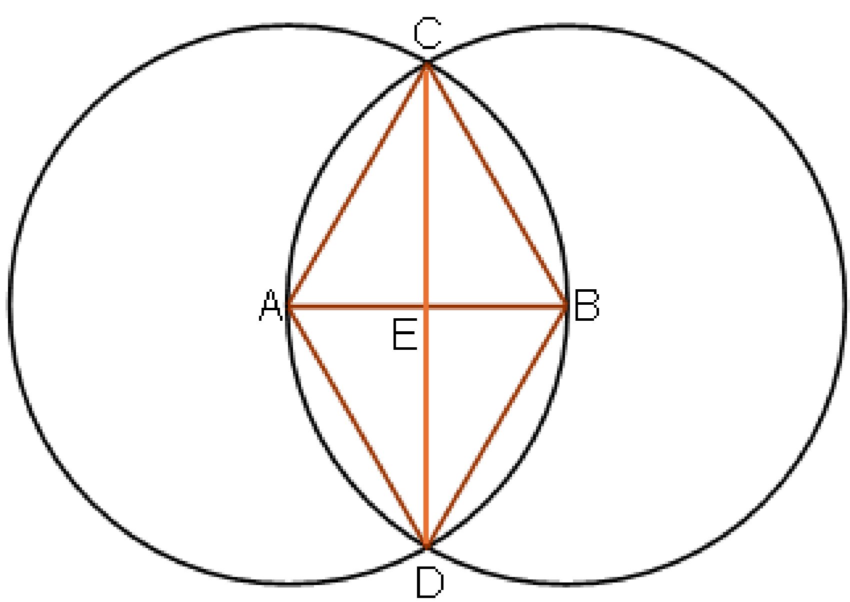


Next, draw a line connecting points C and D as shown, giving us point E. We have now created some additional triangles, including ACE, BCE, ADE and BDE. We have also created triangles ACD and BCD.



In the two larger triangles ACD and BCD we can see that sides AC and CB are equal because they are both sides of an equilateral triangle. Sides AD and BD are equal for the same reason. Side CD is common to both triangles giving us two triangles with all three corresponding sides equal. Therefore, they are identical triangles having all angles equal as well; wherefore the two angles ACE and ECB are equal and the two angles ADE and BDE are equal.

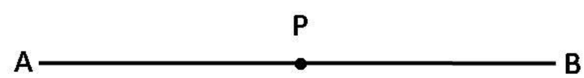
In the smaller triangles ACE and BCE sides AC and CB are equal, side CE is common to both and angles ACE and BCE are equal, as demonstrated, therefore we have the SAS proposition satisfied, which means that triangles ACE and BCE are identical; wherefore sides AE and BE are equal. In this manner we can show that line AB is bisected. It also follows that the triangles ACE and BCE being identical, the adjacent angles AEC and BEC are equal, and are therefore, by definition, right angles, which makes the lines AB and CD perpendicular.



So, in this one exercise we have the ability to bisect a line, and a means of creating equilateral triangles, right angles and perpendicular lines, as well as the entire edifice of Euclidian geometry.

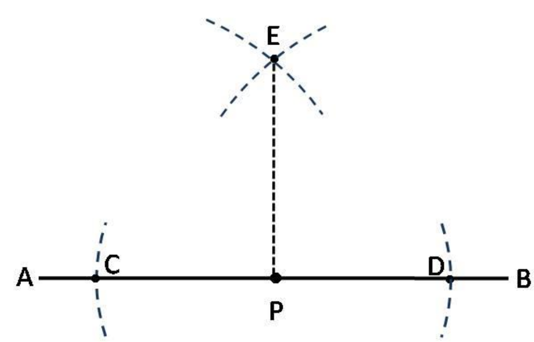
Proposition 8

From a given point on a straight line to draw a perpendicular to that line.



Given line AB and point P on the line

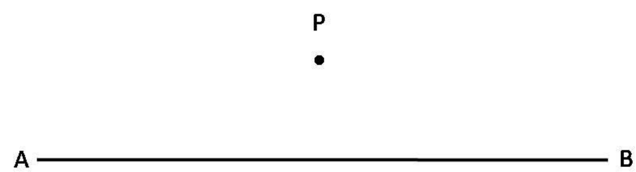
Using P as a center and any convenient radius, draw circular arcs cutting line AB in two points C and D as shown. Using C and D as centers open compass to a radius somewhat greater than half the distance from C to D and draw portions of two arcs of equal radii whose intersection gives point E.



Laying a straight edge from points E to P draw a line of any appropriate length. That line, EP, will be perpendicular to line AB. The two arcs giving point E can be drawn to intersect either above or below line AB.

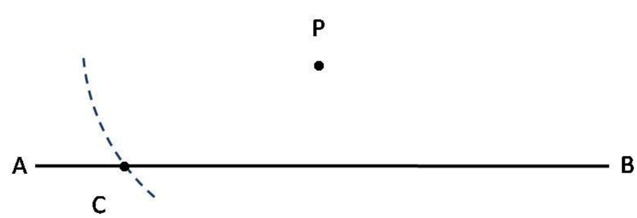
Proposition 21

Through a given point draw a line parallel to a given line.

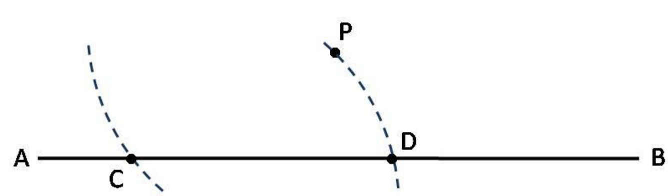


Given line AB and point P not on the line

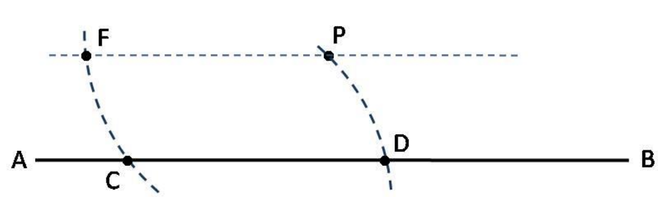
Using point P as a center draw an arc of any convenient radius cutting line AB producing point C. The arc must be of a length greater than the distance separating point P from the line AB, as shown



Using point C as a center draw an arc with radius CP intersecting line AB so as to produce point D.

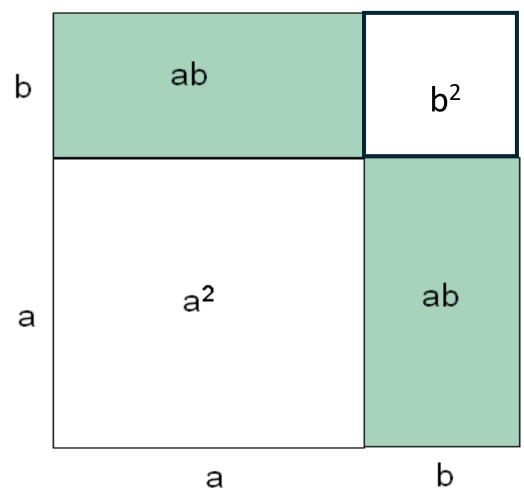


Set the compass center at D and adjust the radius to DP. Set the compass on point C and transfer the distance DP to the arc through C so as to produce point F.



Lay the straight edge across points F and P and draw a line of any appropriate length. This line will be parallel to line AB.

We can visualize completing the square in the geometric sense by studying the following diagram.



We can consider the area of the square from two vantage points: geometric and algebraic. Geometrically speaking we can study the figure as a whole and as the sum of its parts. You can see that the total area is the sum of the four separate areas a^2, ab, ab and b^2. Each area value of the four parts, the two squares and the two rectangles, is simply the product of the two side lengths. Written as an expression the total area would be

$$a^2 + ab + ab + b^2 \quad \text{or} \quad a^2 + 2ab + b^2$$

You can also think of the total area as one side length of the composite figure multiplied by the other side length, that is, the side length squared. You can easily see that the length of either side is a + b. Written out as an expression the area would be

$$(a + b)\,(a + b)$$

FOILing this we get

$$a^2 + ab + ab + b^2$$

Combining terms we have

$$a^2 + 2ab + b^2$$

Which is what we had when we looked at the whole square geometrically as the sum of its' parts. The form above is referred to as a perfect square trinomial, and as you have seen, when you take the square root of the expression such as this, you simply take the square root of the first term of the trinomial and the square root of the last term conjoined by the sign of the middle term.

By using the methods of algebra without any consideration of a geometric figure we can work our way from this:

$$(a + b)^2$$

to this: $a^2 + 2ab + b^2$

and back again.

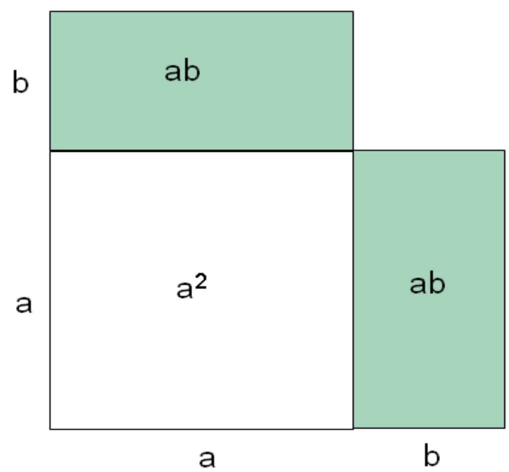
By learning to recognize the pattern of a perfect square trinomial you will have at your disposal a powerful tool for extracting square roots from quadratic equations. Think of it this way: Whenever you square a binomial you get a perfect square trinomial, so, whenever you are faced with a perfect square trinomial you can "unsquare" it immediately because the square root of the thing is a simple binomial whose two terms are the square root of the first term and the square root of the last term of the trinomial.

However, in solving real world problems that involve a squared term (which means solving a quadratic equation) it is not always going to be the case that it can automatically be represented by a perfect square trinomial.

You will remember that, in the geometric sense, a perfect square trinomial is an algebraic depiction of a perfect square figure that can be represented as the sum of its parts.

But what if one of the parts is missing?

Many problems in mathematics that must be solved can be thought of as the equivalent of the following diagram, in which the small part b^2 is missing.



You can immediately see that if one were to add the missing piece to the upper right corner you would have a perfect square whose square root could easily be acquired as it would be the side length of the completed square, which is nothing more than the simple binomial a + b. As it stands in its' incomplete state the area of the figure would be represented as

$$a^2 + 2ab$$

To "complete the square" you need an easy way to supply the missing piece. As it turns out there is an easy way.

Before we get to that let's briefly ponder this situation in the context of a problem to be solved. Using mathematics to solve problems is basically knowing how to set up the parameters of the problem in an equation. If the equation involves a term squared it is a "quadratic equation." Many different types of problems require dealing with quadratics; therefore, to solve these types of problems one must know how to solve quadratic equations. And this usually means that somewhere in the process of "isolating the variable" you must get the squared term out of the equation.

Completing the Square provides the key to accomplishing this.

In terms of our diagram above, what is typically the case is that the $a^2 \pm 2ab$ will be on one side of the equation with something else on the other side. So once having determined the value of b^2 we need only add it to both sides of our equation, which having been done turns $a^2 \pm 2ab$ into the perfect square trinomial $a^2 \pm 2ab + b^2$, from which we can quickly and easily extract the square root. The trick is knowing what value to add to the incomplete expression to convert it into a perfect square trinomial, in other words, how to determine the value of b^2.

As it turns out, there is a consistent relationship between the last term of any perfect square trinomial and the coefficient of the middle term.

The method can be readily understood by taking some examples.

Let's start by squaring some simple binomials:

$(x + 3)^2 = (x + 3)(x + 3) = x^2 + 6x + 9$

$(q + 5)^2 = (q + 5)(q + 5) = q^2 + 10q + 25$

$(r - 7)^2 = (r - 7)(r - 7) = r^2 - 14r + 49$

$(w + 9)^2 = (w + 9)(w + 9) = w^2 + 18w + 81$

In each case what results from the action of squaring is a perfect square trinomial. Can you discern the relationship between the middle term of the trinomial and the final constant term? Look at the coefficient of the middle term. Do you see that in each case above, the square root of the final constant term doubled gives you the coefficient of the middle term? Conversely, by taking half of the said coefficient and squaring it you get the correct final term, which must be added in order to transform the incomplete expression into a perfect square trinomial. So, if you can change one side of a quadratic equation to produce a perfect square trinomial you can then take the square root of both sides, thereby eliminating the squared term from that side.

Construction of the Pentagon

Central to the construction of the regular 5-sided polygon known as the pentagon is the employment of the Golden Section and the 1 – 2- √5 triangle from which it derives.
Now is the time to recall the right triangle with legs in the ratio of two to one and hypotenuse equal to the Square Root of Five.

Let us suppose we have that very triangle and we decide to arrange it so that it stands on its' short side as a base as you see to the right. I should mention that typically, in creating a pentagon, one would begin with the circle, but we will approach it this way, better to emphasize the relation of the pentagon with this Root 5 triangle and the Golden Section.

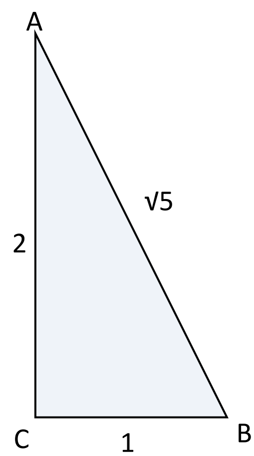


Figure 1

The procedure we are about to perform to draw a pentagon would work just as well were we to use the long leg as the base. However, that would have the consequence that a vertex would be pointing down rather than up. Maybe it's an aesthetic thing but I prefer my pentagons to stand on a flat side as a base. After drawing your pentagon you will be clear on this.

For our demonstration we will scale down Figure 1 to ¾ its size above, keeping the side length ratios and labeling the same. Placing your compass at point C, draw a full circle with radius of CA, (or 2). Extend the base CB to the left until it meets the circle forming a horizontal radius CD, also of length 2. Next, using B as a center set your compass to length BA and swing an arc down to radius CD giving you point G. Your drawing should now look like Figure 2.
Since AB equals √5 so does GB. GC then must be equal to GB – CB, or √5 – 1.

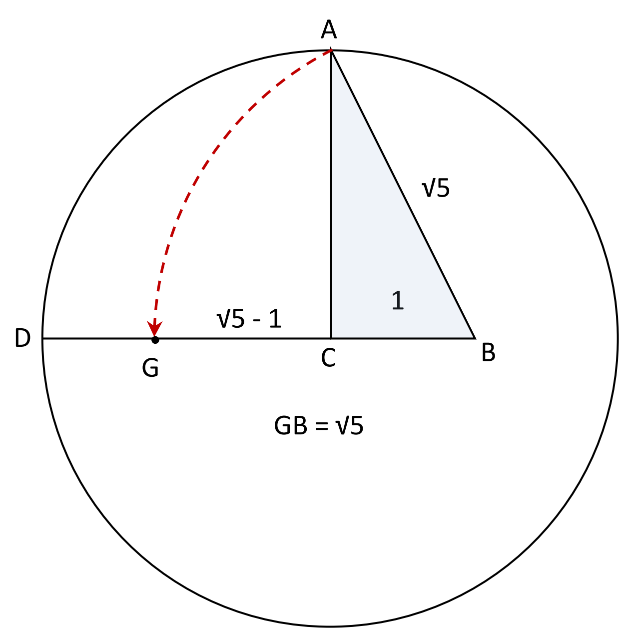


Figure 2

If we were to draw a line GA we would find that it was somewhat longer than √5, since its' base is somewhat longer than 1. (The decimal value of √5 – 1 = 1.236…) By drawing line AG we have formed a new right triangle ACG whose long leg is 2, whose short leg is √5 – 1, and whose hypotenuse is unknown. To further clarify the role of the Golden Section in deriving the pentagon I am going to cut all the sides of ABG in half, giving us side lengths as shown in Figure 3. Of course, this triangle is similar to our original, which means the ratios of corresponding sides are exactly the same, and it's the ratios we are interested in.

Using the Pythagorean Theorem to find the length of AG we have

$$1^2 + \left(\frac{\sqrt{5}-1}{2}\right)^2 = (AG)^2$$

Squaring the both terms on the left:

$$1 + \frac{1-2\sqrt{5}+5}{4} = (AG)^2$$

Combine terms in the fraction we get

$$1 + \frac{6-2\sqrt{5}}{4} = (AG)^2$$

Splitting the fraction transforms it to:

$$1 + \frac{6}{4} - \frac{2\sqrt{5}}{4} = (AG)^2$$

Reducing terms and finding the common denominator:

$$\frac{2}{2} + \frac{3}{2} - \frac{\sqrt{5}}{2} = (AG)^2$$

Adding the fractions gives:

$$\frac{5-\sqrt{5}}{2} = (AG)^2$$

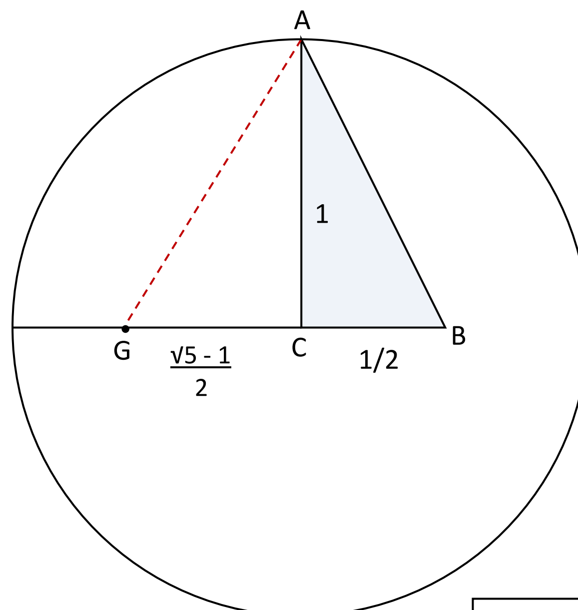


Figure 3

And, finally, when we take the square root of both sides we get the dimension we are looking for:

$$AG = \frac{\sqrt{5-\sqrt{5}}}{\sqrt{2}} = 1.17557\ldots$$

With a line length equal to GA we are now ready to create the pentagon. Simply place your compass at point A and open it to a radius of AG. (which will be slightly larger than your compass setting from the previous step in which it was set to AB, or BG). Using point A as a center, swing an arc in both directions cutting the circle in two points J and K as shown in Figure 4 on the next page.

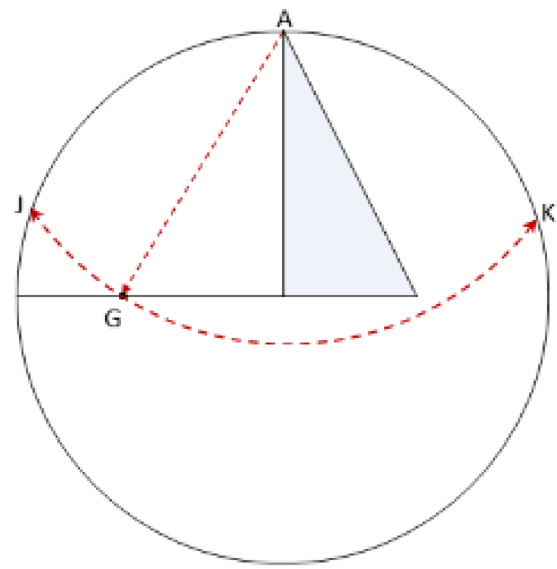


Figure 4

Then, using J and K as centers, and without changing the setting of your compass, strike off two additional points on the circumference giving you points L and M. The five points A, J, K, L and M define the five corners of a perfect pentagon.

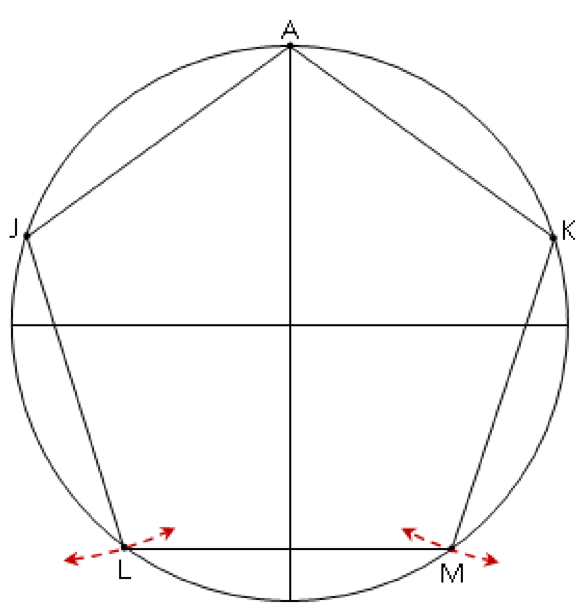
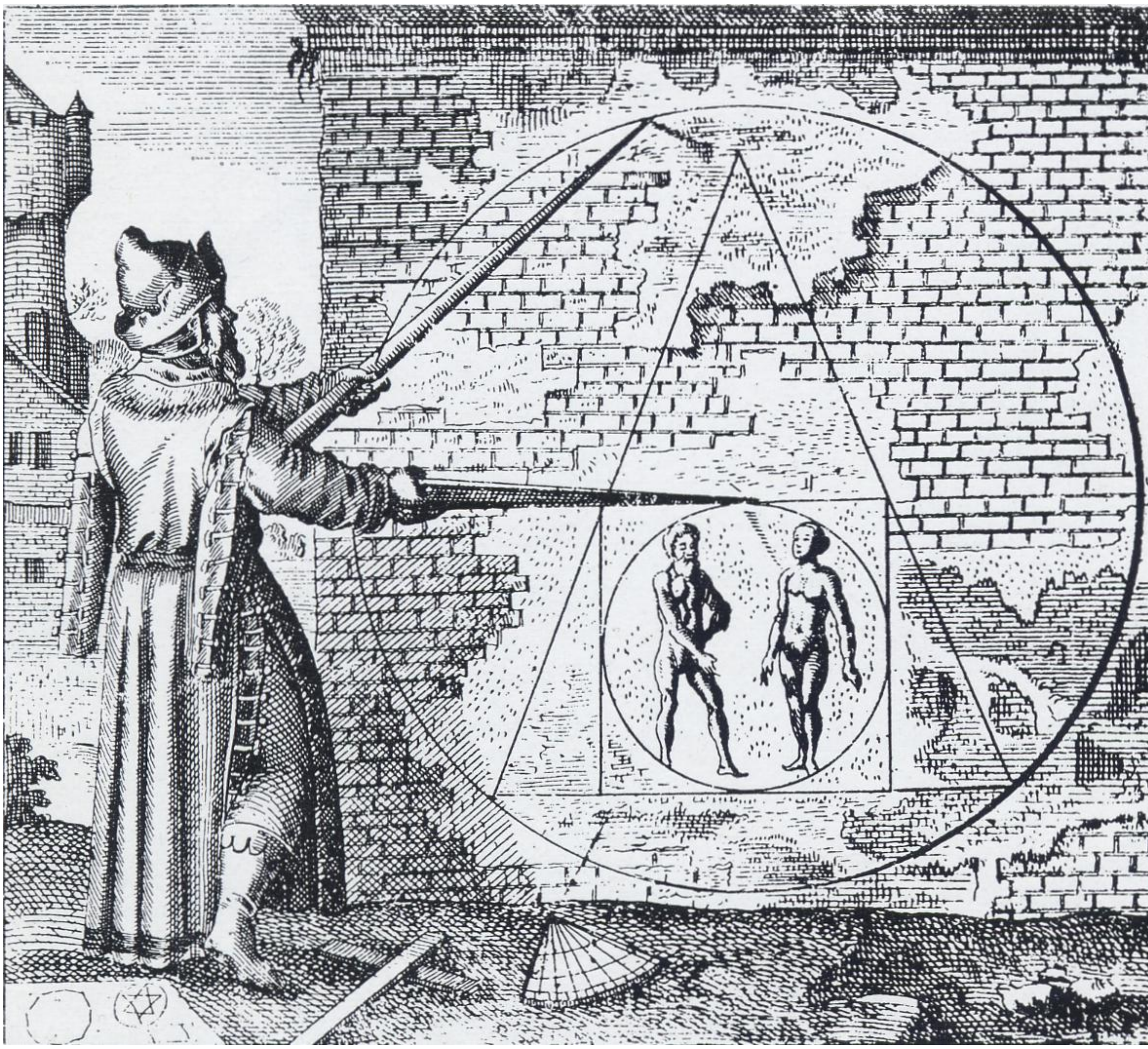


Figure 5

Compared to the generation of Ad Triangulum or Ad Quadratum figures, and the relatively simple geometric logic necessary for their proofs, that of the pentagon is not so straight forward. In other words, how would you prove that the figure above is a perfect pentagon?

Rebis: The Divine Consummation of the Great Work

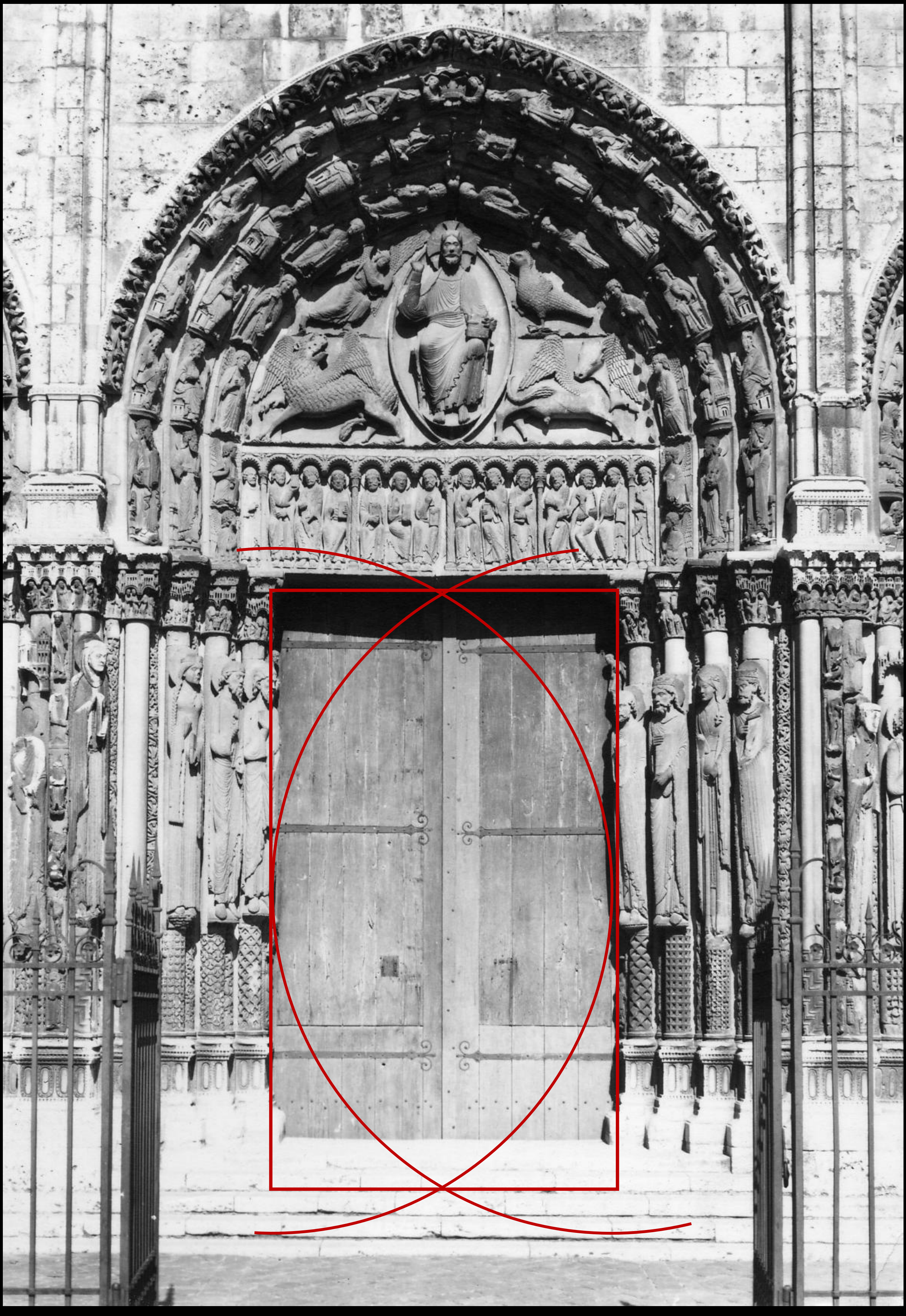


Correct proportion is the only method of breaking down the Wall of Mystery.

Michael Maier: Atalanta Fugiens - 1618

Fu Hsi and Nu Qua: Chinese Creator Gods, 12th century

Fu Hsi holds the square and Nu Qua holds the compass and the plumb line.



West portal of Chartres Cathedral: The Portal of Judgement